Dependent Data in Economic and Financial Problems

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Abstract. Optimization problems depending on probability measures correspond to many economic and financial applications. The paper deals with the case when an empirical measure substitutes the theoretical one. Especially, the paper deals with a convergence rate of the corresponding estimates. "Classical" results for independent samples are recalled, situations in which the case of dependent sample can be (from the mathematical point of view) reduced to independent case are mentioned. A great attention is paid to weak dependent samples fulfilling the Φ–mixing condition.

Keywords: Stochastic programming, Wasserstein metric, $\mathcal{L}_1$ norm, Empirical estimates, One–stage problems, Multistage problems, Independent samples, m–dependent sequences, Markov dependence, Φ– mixing random sample

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1 Introduction

Economic activities are usually simultaneously influenced by a random factor and a decision parameter. Constructing their mathematical models we often obtain optimization problems depending on a probability measure. These models can be static or dynamic. Multistage stochastic problems belong to a dynamic types. Employing a recursive definition we obtain a system of one–stage problems. Consequently, the results obtained for one–stage problems can be often employed to study multistage cases.

1.1 One–Stage Model

Let $\xi \,(:= \xi(\omega) = [\xi_1(\omega), \ldots, \xi_s(\omega)])$ be s–dimensional random vector defined on a probability space $(\Omega, \mathcal{S}, P)$; $F(:= F(z), z \in R^s)$ the distribution function of ξ; F_i, $i = 1, \ldots, s$ one–dimensional marginal distribution functions corresponding to F; P_F, Z_F the probability measure and support corresponding to F. Let, moreover, $g_0(:= g_0(x, z))$ be a real–valued function defined on $R^n \times R^s$; $X \subset R^n$ be a nonempty set. If the symbol E_F denotes the operator of mathematical expectation corresponding to F, then a rather general "classical" one–stage stochastic programming problem can be introduced in the form:

Find
$$\varphi(F) = \inf\{\mathsf{E}_F g_0(x, \xi) | x \in X\}. \tag{1}$$

In applications very often the "underlying" probability measure P_F has to be replaced by an empirical one. Evidently, then the solution is sought w.r.t. the "empirical problem":

Find
$$\varphi(F^N) = \inf\{\mathsf{E}_{F^N} g_0(x, \xi) | x \in X\}, \tag{2}$$

where F^N denotes an empirical distribution function determined by a random sample $\{\xi^i\}_{i=1}^N$ corresponding to the distribution function F. If we denote the optimal solutions sets of (1) and (2) by $\mathcal{X}(F)$, $\mathcal{X}(F^N)$, then $\varphi(F^N)$, $\mathcal{X}(F^N)$ are stochastic estimates of $\varphi(F)$, $\mathcal{X}(F)$.

The investigation of these estimates started in 1974 by R. Wets (see [27]). In the same time consistency has been investigated under ergodic assumption in [10]. These papers have been followed many times (see e.g. by [5], [19], [20], [22]). The investigation of the convergence rate started in [11], and followed e.g. by [2], [8], [21], [25], [26]. The investigation for weakly dependent samples started in eighties (see [12]) and followed e.g. by [2], [8], [13], [14].

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1.2 Multistage Model

A general multistage stochastic programming problem can be in a rather general form introduced recursively (see e.g. [6], [15]) as the problem:

Find

$$\varphi_{\mathcal{F}}(M) = \inf \{\mathsf{E}_{F^{\zeta^0}} g^0_{\mathcal{F}}(x^0, \zeta^0) | \; x^0 \in \mathcal{K}^0\}, \tag{3}$$

where $g^0_{\mathcal{F}}(x^0, z^0)$ is given recursively

$$\begin{aligned} g^k_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) \quad &= \quad \inf\{\mathsf{E}_{F^{\zeta^{k+1}|\bar{\zeta}^k = \bar{z}^k}} \; g^{k+1}_{\mathcal{F}}(\bar{x}^{k+1}, \bar{\zeta}^{k+1}) \,|\, x^{k+1} \in \mathcal{K}^{k+1}(\bar{x}^k, \bar{z}^k)\}, \quad k = 0, 1, \ldots, M-1, \\ g^M_{\mathcal{F}}(\bar{x}^M, \bar{z}^M) \quad &:= \quad g^M_0(\bar{x}^M, \bar{z}^M). \end{aligned} \tag{4}$$

$\zeta^j = \zeta^j(\omega)$, $j = 0, 1, \ldots, M$ are s–dimensional random vectors defined on a probability space (Ω, S, P); $\bar{\zeta}^k = \bar{\zeta}^k(\omega) = [\zeta^0, \ldots, \zeta^k]$, $\bar{z}^k = [z^0, \ldots, z^k]$, $z^j \in R^s$, $x^j \in R^n$, $\bar{x}^k = [x^0, \ldots, x^k]$, $j = 0, 1, \ldots, k$, $k = 0, 1, \ldots, M$; $F^{\zeta^j}(z^j)$, $F^{\bar{\zeta}^j}(\bar{z}^j)$, $j = 0, \ldots, M$ the distribution functions of the ζ^j and $\bar{\zeta}^j$; $F^{\zeta^k|\bar{\zeta}^{k-1}}(z^k| \bar{z}^{k-1})$, $k = 1, \ldots, M$ denotes the conditional distribution function (ζ^k conditioned by $\bar{\zeta}^{k-1}$). $g^M_0(\bar{x}^M, \bar{z}^M)$ is a function defined on $R^{n(M+1)} \times R^{s(M+1)}$; $\mathcal{K}^{k+1}(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \ldots, M-1$, are multifunction mappings $R^{n\,(k+1)} \times R^{s\,(k+1)}$ into the space of (mostly compact) subsets of $\mathcal{X}$; $\mathcal{X}$, $\mathcal{K}^0 \subset R^n$ are nonempty sets; $\mathcal{K}^0 \subset \mathcal{X}$. $Z^j_{\mathcal{F}} \subset R^s$, $j = 0, 1, \ldots, M$, denote the supports corresponding to $F^{\xi^j}(\cdot)$; $\bar{Z}^k_{\mathcal{F}} = Z^0_{\mathcal{F}} \times \ldots \times Z^k_{\mathcal{F}}$, $\bar{\mathcal{X}}^k = \mathcal{X} \times \ldots \times \mathcal{X}$, $k = 0, 1, \ldots, M$.

Evidently, the problem given by (3) and (4) is depending on the system:

$$\mathcal{F} = \{F^{\zeta^0}(z^0), \quad F^{\zeta^k|\bar{\zeta}^{k-1}}(z^k|\bar{z}^{k-1}), \, k = 1, \ldots, M\}. \tag{5}$$

2 Historical Survey

First we recall "classical" results on consistency and convergence rate.

Theorem 1. [10]. If

1. X is a compact set, $g_0(x, z)$ is a uniformly continuous bounded function on $R^s \times X$,
2. $\{\xi^i\}_{i=1}^N$, $N = 1, 2, \ldots$ is a random sample corresponding to ergodic sequence,

then

$$P\{\omega : |\varphi(F^N) - \varphi(F)| \xrightarrow[N \to \infty]{} 0\} = 1.$$

(The ergodicity has been considered in the sense of [1].)

Theorem 2. Let X be a nonempty compact set. If

1. in every $x \in X$ the function $g_0(x, \xi)$ is a continuous function of x for almost every $\xi \in Z_F$,
2. $g_0(x, \xi), x \in X$ is dominated by an integrable function,
3. $\{\xi^i\}_{i=1}^N, N = 1, 2, \ldots$ is independent random sample,

then

$$P\{\omega : |\varphi(F^N) - \varphi(F)| \xrightarrow[N \to \infty]{} 0\} = 1.$$

Proof. The assertion of Theorem 2 follows from Proposition 5.2 and Theorem 7.48 proven in [25].

Theorem 3. [11] Let $t > 0$, X be a nonempty compact, convex set. If

1. $g_0(x, z)$ is a uniformly continuous function on $X \times Z_F$ bounded by $M > 0$ (i. e., $|g_0(x, z)| \le M$),
2. $g_0(x, z)$ is a Lipschitz function on X with the Lipschitz constant L',
3. $\{\xi^i\}_{i=1}^N$, $N = 1, 2, \ldots$ is an independent random sample,

then there exist $K(t, X, L') > 0$ and $k_1(M) > 0$ such that

$$P\{\omega : |\varphi(F) - \varphi(F^N)| > t\} \le K(t, X, L') \exp\{-N k_1(M) t^2\}.$$

Remark. The assertion of Theorem 3 is valid independently of the distribution function F; consequently also for the distribution functions with heavy tails. On the other hand g_0 must be a bounded function. Moreover under the assumptions of Theorem 3 it has been proven in [13] that

$$P\{\omega : N^{\beta}|\varphi(F) - \varphi(F^N)| > t\} \xrightarrow[N \longrightarrow \infty]{} 0 \quad \text{for} \quad \beta \in (0, \tfrac{1}{2}).$$

If the moment generating function $M_{g_0}(t)$, corresponding to $g_0(x, \xi)$, is defined by the relation $M_{g_0}(t) := \mathsf{E}_F\{e^{t[g_0(x,\xi) - \mathsf{E}_F g_0(x,\xi)]}\}$, then we can recall the following assertion.

Theorem 4. [24] Let $X \subset R^n$ be a nonempty closed set, $\| \cdot \| = \| \cdot \|_n^2$ denotes the Euclidean norm in R^n. If

1. for $x \in X$ the moment generating function $M_{g_0}(t)$ is finite valued for all t in a neighbourhood of zero,
2. there exists a measurable function $\kappa \colon Z_F \to R_+$, and a constant $\gamma > 0$ such that
$$|g_0(x', \xi) - g_0(x, \xi)| \le \kappa(\xi) \|x' - x\|^\gamma \quad \text{for all} \quad \xi \in Z_F,\, x, x' \in X,$$
3. the moment generating function $M_\kappa(t)$ of $\kappa(\xi)$ is finite valued for all t in a neighbourhood of zero,

then for any $\varepsilon > 0$ there exist positive constants $C = C(\varepsilon)$ and $\beta = \beta(\varepsilon)$, independent of N, such that
$$P\{\sup_{x \in X} |\mathsf{E}_{F^N} g_0(x, \xi) - \mathsf{E}_F g_0(x, \xi)| \ge \varepsilon\} \le C(\varepsilon) e^{-N\beta(\varepsilon)}.$$

3 Wasserstein Metric via Empirical Estimates

Let $\mathcal{P}(R^s)$ denote the set of Borel probability measures on $R^s, s \ge 1$, $\| \cdot \|_s^1$denote $\mathcal{L}_1$ norm in R^s and $\mathcal{M}_1(R^s) = \{P \in \mathcal{P}(R^s) : \int\limits_{R^s} \|z\|_s^1 P(dz) < \infty\}$. We introduce the following system of the assumptions:

A.1 • $g_0(x, z)$ is a uniformly continuous function on $X \times R^s$,
- $g_0(x, z)$ is for $x \in X$ a Lipschitz function of $z \in R^s$ with the Lipschitz constant L (corresponding to the $\mathcal{L}_1$ norm) not depending on x,

A.2 • $\{\xi^i\}_{i=1}^\infty$ is independent random sequence corresponding to F,
- F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^N, N = 1, 2, \ldots,$

A.3 $P_{F_i}, i = 1, \ldots, s$ are absolutely continuous w. r. t. the Lebesgue measure on R^1; F_i, f_i, P_{F_i}, $i = 1, 2, \ldots, s$ denote one-dimensional marginal distribution function, probability density and the probability measure corresponding to F.

Proposition 1. [16] Let $P_F, P_G \in \mathcal{M}_1(R^s)$, and X be a compact set. If A.1 is fulfilled, then
$$|\varphi(F) - \varphi(G)| \le L \sum_{i=1}^s \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i.$$

Replacing G by F^N in Proposition 1 and supposing $s = 1$ we can obtain for the random value $\int\limits_{-\infty}^{+\infty} |F(z) - F^N(z)| dz$ the following result.

Proposition 2. [19] Let $s = 1, t > 0$ and A.2, A.3 be fulfilled. Let, moreover, $\mathcal{N}$ denote the set of natural numbers. If there exists $\beta > 0$, $R := R(N) > 0$ defined on $\mathcal{N}$, $R(N) \xrightarrow[N\to\infty]{} \infty$ and, moreover,
$$\begin{aligned} & N^\beta \int\limits_{-\infty}^{-R(N)} F(z) dz \xrightarrow[N\to\infty]{} 0, \qquad N^\beta \int\limits_{R(N)}^{\infty} [1 - F(z)] dz \xrightarrow[N\to\infty]{} 0, \\ & 2NF(-R(N)) \xrightarrow[N\to\infty]{} 0, \qquad 2N[1 - F(R(N))] \xrightarrow[N\to\infty]{} 0, \\ & (\tfrac{12N^\beta R(N)}{t} + 1) \exp\{-2N(\tfrac{t}{12R(N)N^\beta})^2\} \xrightarrow[N\longrightarrow\infty]{} 0, \end{aligned} \tag{6}$$
then
$$P\{\omega : N^\beta \int_{-\infty}^{\infty} |F(z) - F^N(z)| dz > t\} \xrightarrow[N\to\infty]{} 0. \tag{7}$$

4 Convergence Rate

4.1 One–Stage Problem: Independent Sample

Theorem 4. [19] Let $t > 0$, $\beta \in (0, \frac{1}{2})$, A.1, A.2 and A.3 be fulfilled. Let, moreover, $\{\xi^i\}_{i=1}^\infty$ be independent random sequence corresponding to F. If there exists constants C_1, C_2 and $T > 0$ such that
$$f_i(z_i) \le C_1 \exp\{-C_2|z_i|\} \quad \text{for } z_i \in (-\infty, -T) \cup (T, \infty) \quad \text{and} \quad i = 1,\, 2,\, \ldots,\, s,$$

then

$$P\{\omega : N^{\beta}|\varphi(F^N) - \varphi(F)| > t\} \xrightarrow[N\to\infty]{} 0.$$

4.2 One–Stage Problem: m–Dependent Random Sample

To recall a definition of m– sequences, let $\{\xi^i\}_{i=-\infty}^{\infty}$ be a strictly stationary s–dimensional random vectors. We denote by the symbol $\mathcal{F}_c^d$ the σ–algebra generated by ξ^i, $c \le i \le d$.

Definition 1. [4] $\{\xi^i\}_{i=-\infty}^{\infty}$ is said to be m–dependent sequence ($m \ge 2$) if $\mathcal{F}_{-\infty}^{a}$ and $\mathcal{F}_b^{\infty}$ are independent for $b - a > m$.

Theorem 5. [17] Let $t > 0$, $\{\xi^i\}_{i=1}^{\infty}$ be a strictly stationary m sequence of s–dimensional random vectors corresponding to distribution function F. If A.1, A.3 are fulfilled and if there exist C_1, C_2, $T > 0$ such that

$$f_i(z_i) \le C_1 \exp\{-C_2|z_i|\} \quad \text{for} \quad z_i \in (-\infty, -T) \bigcup (T, \infty), \quad i = 1, \ldots, s,$$

then

$$P\{\omega :\ N^{\beta}|\varphi(F^N) - \varphi(F)| > t\} \longrightarrow_{(N\longrightarrow\infty)} 0 \quad \text{for } \beta \in (0, \frac{1}{2}).$$

4.3 Multistage Problem –Markov Dependence

To investigate problems (3) and (4) we restrict to the case when $\{\zeta^j\}_{j=-\infty}^{\infty}$ fulfils the Markov type of dependence and recall that $\{\zeta^j\}_{j=0}^{\infty}$ corresponds to a homogenous Markov chains iff ζ^j, $j = 0, \ldots,$ can be represented by a recurrence equation $\zeta^j = \bar{H}(\zeta^{j-1}, \varepsilon^j)$, where $\bar{H}$ is a measurable function and ε^j, $j > 0$ is i.i.d. sequence independent of ζ^0 (for more details see [3] or [9]). A (rather general) Markov type dependence has been considered in [15]. We consider only a special case. To this end we assume:

A.4 $\{\zeta^k\}_{k=-\infty}^{\infty}$ follows a (generally) nonlinear autoregressive sequence $\zeta^k = H(\zeta^{k-1}) + \varepsilon^k$, where ζ^0, ε^k, $k = 1, 2, \ldots$ are stochastically independent; ε^k, $k = 1, 2, \ldots$identically distributed. $H := (H_1, \ldots, H_s)$ is a Lipschitz vector function defined on R^s. We denote the distribution function of $\varepsilon^1 = (\varepsilon_1^1, \ldots, \varepsilon_s^1)$ by the symbol F^{ε} and suppose the realization ζ^0 to be known.

If A.4 is fulfilled, then (3), (4) is a s system of one–stage stochastic (mostly parametric) programming problems. Moreover, the system $\mathcal{F}$ is (under A.4) determined by the $P_{F^{\varepsilon}}$ Consequently, empirical estimates $\mathcal{F}^N$ of the $\mathcal{F}$ are determined by i.i.d. $\{\varepsilon^j\}_{j=1}^{N}$, $N = 1, \ldots$ Evidently, the problems (3), (4) is (from the mathematical point of view) transformed to one–stage case.

Theorem 6. [18] If

i.1 $g_{\mathcal{F}}^k(\bar{x}^k, \bar{z}^k)$, $k = 1, \ldots, M$ are for $\bar{x}^k$, $\bar{z}^{k-1}$ a Lipschitz function of z^k with the Lipsch. const. L_k,

i.2 $g_{\mathcal{F}}^0(\bar{x}^0, \bar{z}^0)$ is for $x^0 \in \mathcal{K}_0$ a Lipschitz function of z^0 with the Lipschitz const. L_0,

i.3 $\mathcal{K}^0$, $\mathcal{K}^{k+1}(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \ldots, k-1$ are compact sets,

i.3 $P_{F_i^{\varepsilon}}$, $i = 1, \ldots, s$ are absolutely continuous with respect to Lebesgue measure on R^1 (we denote by f_i^{ε}, $i = 1, \ldots, s$ the corresponding probability densities,

i.4 there exist constants C_1, $C_2 > 0$, $T > 0$ such that
$f_i^{\varepsilon}(z_i) \le C_1 \exp\{-C_2|z_i|\}$ for $z_i \in (-\infty, -T) \bigcup (T, \infty)$, $i = 1, \ldots, s$.

then

$$P\{\omega : N^{\beta}|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{F}^N}(M)| > t\} \longrightarrow_{N\longrightarrow\infty} 0 \quad \text{for} \quad T > 0, \beta \in (0, 1/2).$$

(The conditions under which the assumptions i.1, i.2, i.3 are valid can be found in [15].)

4.4 One–Stage Problem: Φ–Mixing Random Samples

Definition 2. The sequence $\{\xi^i\}_{i=-\infty}^{\infty}$ is called Φ–mixing (uniformly mixing) whenever there exists Φ_N, $\Phi_N \to 0$ as $N \to \infty$ fulfilling the relation

$$|P(A \cap B) - P(A)P(B)| \le \Phi_N P(A), \quad A \in \mathcal{F}_{-\infty}^{k}, \quad B \in \mathcal{F}_{k+N}^{\infty}, \quad -\infty < k < \infty.$$

Remark. [28] (see also [4]) If $\{\xi^i\}_{-i=\infty}^{\infty}$ is fulfilling the conditions of Φ–mixing and simultaneously it is strictly stationary Gaussian sequence, then there exists m such that ξ^i, $i = \ldots, -1, 0, 1 \ldots$ is a m sequence. (The proof of this assertion belongs to Ibragimov ([7]).

Lemma 1. If ξ^i is a uniformly mixing sequence of centered random variables with $|\xi^i| \le 1$ such that $\sum\limits_{N=1}^{\infty} \Phi_N < \infty$; $b^{-1} = (1 + 4 \sum\limits_{N=1}^{\infty} \Phi_N)$, $a = 2 \exp 3\sqrt{e}$ and if N and $t > 0$ satisfy

$$N\sigma^2 \ge 1, \quad 0 \le t \le \frac{\sigma\sqrt{N}}{8bk_N} \quad \text{for} \quad k_N = \inf\{k : \frac{\Phi_k}{k} \le \frac{1}{N}\}, \quad \sigma^2 = \sup_N \mathsf{E}_F |\xi^N|^2, \tag{8}$$

then

$$P\{\omega : |\sum_{i=1}^{N} \zeta^i| \ge t\sqrt{N}\sigma\} \le a \exp\{-bt^2\}. \tag{9}$$

Furthermore, let us assume that $\{\xi^i\}_{i=-\infty}^{\infty}$ is one–dimensional strongly stationary random sequence fulfilling Φ–mixing conditions with coefficients Φ_N. If F is the corresponding one dimensional distribution function, then for every $z \in (-\infty, \infty)$, $i = \ldots, -1, 0, 1 \ldots$

$$\mathsf{E}_F I_{(-\infty, z]}(\xi^i) = F(z), \quad \mathsf{E}_F[I_{(-\infty, z]}(\xi^i) - \mathsf{E}_F I_{(-\infty, z]}(\xi^i)]^2 = F(z)(1-F(z)), \quad \frac{1}{N}\sum_{i=1}^{N} I_{(-\infty, z]}(\xi^i) = F^N(z).$$

Evidently, it is easy to see that for every $z \in R^1$ it is possible to apply the assertion of Lemma 1 to random variable $I_{(-\infty, z]}(\xi^i)$ with $\sigma^2 := \sigma^2(\xi^i(z)) = F(z)(1 - F(z))$. However, admitting unbounded support and setting $R \in R^1, R > 0$ then (according to the properties of the distribution functions) it is easy to see that the relation (8) can be simultaneously for $z \in (R, R)$ fulfilled only for "large" N. Moreover, generally then N fulfilling the relation (8) (simultaneously for $z \in (-R, R)$ converges (generally) to ∞ if R converges to ∞. Analyzing the proof of Proposition 2 it is easy to see that employing the approach of Proposition 1 we have to restrict our consideration to the case of bounded support. The following assertion can be proven.

Theorem 7. Let $t > 0$, $\{\xi^i\}_{i=-\infty}^{\infty}$ be a uniformly mixing sequence of s–dimensional random vectors with a common distribution function F and a mixing coefficient Φ_N such that $\sum\limits_{N=1}^{\infty} \Phi_N < \infty$. Let, moreover, X be a nonempty, compact set, A.1, A.3 be fulfilled. If

1. there exist $U_i > 0$, $i = 1, \ldots, s$ such that the support of P_{F_i} is the interval $\langle -U_i, U_i \rangle$,
2. there exist ϑ_i, ϑ^i, $i = 1, \ldots, s$ such that $\vartheta_i \le f_i(z_i) \le \vartheta^i$, $\quad z_i \in \langle -U_i, U_i \rangle$,
3. the mixing coefficients Φ_N fulfil the relations (8) for $z = \max\limits_i \sigma_i(z_i)$, $z = (z_1, \ldots, z_s$ and $R := R(z)$,

then there exists $\beta_0 \in (0, 1/2)$ such that

$$P\{\omega : N^{\beta}|\varphi(F) - \varphi(F^N)| > t\} \longrightarrow_{N \longrightarrow \infty} 0 \quad \text{for} \quad \beta \in (0, \beta_0).$$

Remark. It is easy to see that the value of β_0 depends on the value of Φ mixing coefficients.

5 Conclusion

The paper deals with empirical estimates in the case of stochastic programming problems. First, a consistency results (including ergodic case) has been recalled. However, the aim of the paper has been to compare the results (concerning convergence rate) obtained under the assumption of independent data and some types of dependent samples. Consequently, again the "classical" results (Theorem 3 and Theorem 4) have been recalled. These results have been followed by Theorem 5 and Theorem 6 in which the the assertions have been proven on the basis of a "transformation" of dependent case to the independent one. At the end a result for Φ–mixing sequences has been introduced. To present a detail proof of this last assertion is over the possibility of this contribution. Summarizing we can constant that the former results (considering the problem (1) and depending samples) published former (e.g. in [14]) has been extended.

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